

A NON-VARYING PHENOMENON WITH AN APPLICATION TO THE WIND-TREE MODEL

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ABSTRACT. We exhibit a non-varying phenomenon for the counting problem of cylinders, weighted by their area, passing through two marked (regular) Weierstrass points of a translation surface in a hyperelliptic connected component $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1, g-1)$, $g > 1$. As an application, we obtain the non-varying phenomenon for the counting problem of (weighted) periodic trajectories on the classical wind-tree model, a billiard in the plane endowed with \mathbb{Z}^2 -periodically located identical rectangular obstacles.

1. INTRODUCTION

A connected component of a stratum of Abelian differentials is said to be *non-varying* if for every Teichmüller curve in that component the sum of (positive) Lyapunov exponents is the same. Such a non-varying phenomenon was observed numerically by M. Kontsevich and A. Zorich along with the initial observations on Lyapunov exponents for the Teichmüller geodesic flow [Ko, KZ97]. Today, there are two types of non-varying results. One for low genus, due to D. Chen and M. Möller [CM], which uses a translation of the problem into algebraic geometry. The other one, for hyperelliptic loci, due to A. Eskin, M. Kontsevich and A. Zorich [EKZ], which is a consequence of their main result relating sum of Lyapunov exponents to Siegel–Veech constants, which, roughly speaking, measure the growth rate of the number of cylinders of bounded length on translation surfaces. In particular, the non-varying phenomenon for the sum of Lyapunov exponents is equivalent to the non-varying of Siegel–Veech constants.

The related counting problem has been widely studied and it is related to many other questions such as the calculation of the volume of strata of normalized translation surfaces [EMZ]. H. Masur [Ma88, Ma90] proved that for every translation surface X , there exist positive constants $c(X)$ and $C(X)$ such that the number $N(X, L)$ of (maximal) cylinders of closed geodesics of length at most L satisfy

$$c(X)L^2 \leq N(X, L) \leq C(X)L^2$$

for large enough L . W. Veech [Ve89] proved that for Veech surfaces there are in fact exact quadratic asymptotics. A. Eskin and H. Masur [EMa] proved that for each ergodic probability measure μ on strata of normalized (area 1) translation surfaces, there is a constant $c(\mu)$ such that for almost every surface X , $N(L, X) \sim c(\mu) \cdot \pi L^2$.

This constant $c(\mu)$ is the Siegel–Veech constant ([EMa]); it is the constant in the Siegel–Veech formula ([EMa]), a Siegel-type formula introduced by W. Veech [Ve92], which can be translated into

$$c(\mu) = \frac{1}{\pi R^2} \int N(X, R) d\mu(X).$$

The first explicit computation where made by W. Veech [Ve89, Ve92]. A. Eskin, H. Masur and A. Zorich [EMZ] computed the Siegel–Veech constants for connected components of all strata of Abelian differentials, and also described all possible configurations of cylinders of closed geodesics which might be found on a generic flat surface. In general, the particular constants for Veech surfaces do not coincide with the Siegel–Veech constants of the strata where they live. Unless, of course, we face a non-varying phenomenon, as is for example the case of the hyperelliptic components $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1, g-1)$, $g > 1$.

In this work we study a different but related counting problem: that of cylinders whose core curve passes through two marked regular Weierstrass points on hyperelliptic surfaces in a hyperelliptic component; and we prove the following non-varying phenomenon analogous to the one described above.

Theorem 1. *Let μ be the affine invariant measure supported on the $\mathrm{SL}(2, \mathbb{R})$ -orbit closure of an hyperelliptic surface X in a hyperelliptic component $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1, g-1)$, $g > 1$. Then, the (area) Siegel–Veech constant associated to the counting problem of cylinders whose core curve passes through two marked regular Weierstrass points equals*

$$\begin{cases} \frac{1}{\pi^2} \cdot \frac{1}{2g-1}, & \text{if } X \in \mathcal{H}^{hyp}(2g-2), \\ \frac{1}{\pi^2} \cdot \frac{1}{2g}, & \text{if } X \in \mathcal{H}^{hyp}(g-1, g-1). \end{cases}$$

It is a natural question whether this non-varying phenomenon takes place in every hyperelliptic loci as well, as is the case for the counting problem of every cylinder (and not only those that pass through prescribed Weierstrass points). We shall see that this is not true in general.

The main motivation of this result, is an application to the wind-tree model.

Wind-tree model. The wind-tree model corresponds to a billiard in the plane endowed with \mathbb{Z}^2 -periodic obstacles of rectangular shape aligned along the lattice, as in Figure 1. Denote by $\Pi(a, b)$ the wind-tree model whose obstacles have dimensions $(a, b) \in]0, 1[$.

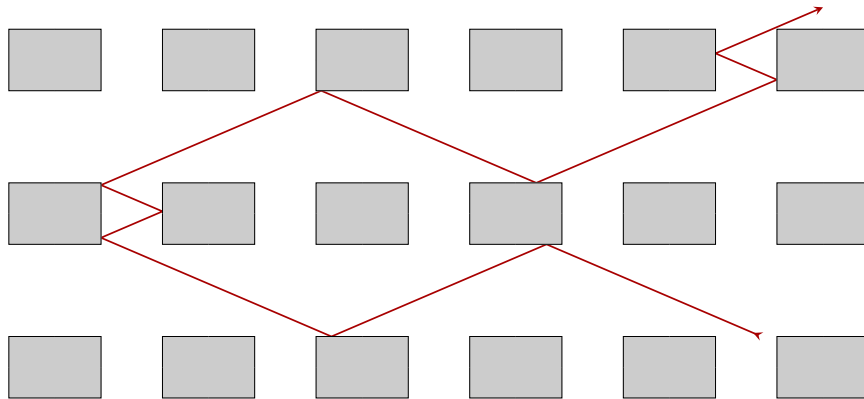


FIGURE 1. The wind-tree model.

The wind-tree model (in a slightly different version) was introduced by P. Ehrenfest and T. Ehrenfest [EE] in 1912. J. Hardy and J. Weber [HW] studied the periodic version. All these studies had physical motivations.

Several advances on the dynamical properties of the billiard flow in the wind-tree model were obtained using geometric and dynamical properties on moduli space of (compact) translation surfaces. A. Avila and P. Hubert [AH] showed that for all parameters of the obstacle and for almost all directions, the trajectories are recurrent. There are examples of divergent trajectories constructed by V. Delecroix [De]. The non-ergodicity was proved by K. Frąćek and C. Ulcigrai [FU]. It was proved by V. Delecroix, P. Hubert and S. Lelièvre [DHL] that the diffusion rate is independent either on the concrete values of the parameters of the obstacle or on almost any direction and almost any starting point and is equals to $2/3$. A generalization of this last result was shown by V. Delecroix and A. Zorich [DZ] for more complicated obstacles.

The result of V. Delecroix, P. Hubert and S. Lelièvre about the diffusion rate *evinces* a first non-varying phenomenon in the case of the classical wind-tree model, which corresponds to the ‘sum of Lyapunov exponents’ counterpart. In this work we describe the ‘Siegel–Veech constant’ counterpart of the non-varying phenomenon.

The author [Pa] studied the counting problem on wind-tree models proving that the number of periodic trajectories has quadratic asymptotic growth rate and computed, in the generic case, the Siegel–Veech constants for the classical wind-tree model as well as for the Delecroix–Zorich variant. In this work we prove that, for the classical wind-tree model, this constant does not depend on the dimensions of the obstacles, exhibiting a non-varying phenomenon analogous to the one described above. More precisely, as a direct consequence of Theorem 1, we have the following.

Theorem 2. *Denote by $N_{area}(\Pi(a, b), L)$ be the number of maximal families of isotopic periodic trajectories (up to \mathbb{Z}^2 -translations) of length at most L in $\Pi(a, b)$, weighted by the area covered by the family.*

- (1) *For Lebesgue-almost every $(a, b) \in]0, 1[$ and, in particular, if a, b are rational or can be written as $1/(1-a) = x + z\sqrt{D}$ and $1/(1-b) = y + z\sqrt{D}$ with $x, y, z \in \mathbb{Q}$ and $x + y = 1$ and D a positive square-free integer, then,*

$$N_{area}(\Pi(a, b), L) \sim \frac{4}{3\pi^2} \cdot \frac{\pi L^2}{1-ab}.$$

- (2) *In any other case, we have the weak asymptotic formula*

$$N_{area}(\Pi(a, b), L) \sim \frac{4}{3\pi^2} \cdot \frac{\pi L^2}{1-ab}.$$

Proof. The statement is a compilation of several different results and is equivalent to say that $c_{area}(\Pi(a, b)) = 4/3\pi^2$ (cf. [AEZ, Theorem 1.7] and [Pa, Theorem 1.2]). By [Pa, Corollary 5.6], the counting problem on the wind-tree model coincides with the counting problem of cylinders whose core curve passes through two marked regular Weierstrass points on a surface $L(a, b) \in \tilde{\mathcal{Q}}(1, -1^5) = \mathcal{H}(2)$.

By elementary considerations on the Siegel–Veech formula (cf. [EKZ, Lemma 1.1]) combined with the lifting properties of cylinders in $L(a, b)$ (see for example [AH, Lemma 3]), we have that $c_{area}(\Pi(a, b))$ is four times the Siegel–Veech constant associated to the corresponding counting on $L(a, b)$.

Thus, by Theorem 1, we conclude that $c_{area}(\Pi(a, b)) = 4/3\pi^2$. \square

Strategy of the proof. From a hyperelliptic surface X in a hyperelliptic component $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1, g-1)$, $g > 1$, and given two fixed regular Weierstrass points, we build three different translation surfaces which are covering of the original surface X . These coverings turn out to be hyperelliptic surfaces as well. We introduce some configurations of cylinders associated to the monodromy of these coverings and describe the counting of cylinders whose core curve passes through the two Weierstrass points in terms of one of these configurations. We relate the Siegel–Veech constants of the configurations on X to their liftings on the coverings. Decomposing the Siegel–Veech constants of the involved surfaces in terms of these configurations, we obtain a system of equations which allows us to describe the Siegel–Veech constants of the configurations in terms of those of the surfaces. Since the surfaces are hyperelliptic, thanks to Eskin–Kontsevich–Zorich [EKZ], the result is non-varying. Describing the hyperelliptic loci where the surfaces lie and putting the values of the corresponding Siegel–Veech constants in the expression allows us to compute explicitly the value of the Siegel–Veech constant associated to the configurations and therefore, the one associated to the counting of cylinders whose core curve passes through the two Weierstrass points.

We present a family of counterexamples for hyperelliptic loci which are not hyperelliptic components. We exhibit hyperelliptic surfaces where the Siegel–Veech constant associated to the counting of cylinders whose core curve passes through two marked Weierstrass points does not coincide with the corresponding Siegel–Veech constant on the hyperelliptic loci where they lie. For this, we use one of the covers defined above, which lies in a hyperelliptic locus which is not a hyperelliptic component. We relate the configuration of cylinders whose core curve passes through (any) two Weierstrass points to one of the configurations mentioned above and we compute the value of the corresponding Siegel–Veech constant analogously. Using a result of Athreya–Eskin–Zorich [AEZ], we show the corresponding generic value for the hyperelliptic locus, which does not coincide with the one obtained for the constructed surface, showing that the relevant Siegel–Veech constant vary along the hyperelliptic locus.

Structure of the paper. In §2 we briefly recall all the background necessary to formulate and prove the results. In §3 we prove the result in the case of the hyperelliptic component $\mathcal{H}^{hyp}(2g-2)$, $g > 1$. We describe the covering construction in §3.1 and prove that they are hyperelliptic surfaces, giving also the corresponding hyperelliptic loci where they lie. In §3.2 we introduce the associated configurations of cylinders and relate the counting of cylinders whose core curve passes through the two Weierstrass points in terms of one of these configurations. We describe the system of equations they satisfy and find the value of the desired Siegel–Veech constant. In §4 we prove the result in the case of the hyperelliptic component $\mathcal{H}^{hyp}(g-1, g-1)$, $g > 1$, following the same outline.

We present in §5 the family of counterexamples, providing the values of (the sum of) the pertinent Siegel–Veech constants for the counterexamples as well as for the generic case.

2. BACKGROUND

2.1. Flat surfaces. For an introduction and general references to this subject, we refer the reader to the surveys of Zorich [Zo], Forni–Matheus [FM], Wright [Wr].

2.1.1. Flat surfaces and strata. Let $g \geq 1$, $\{n_1, \dots, n_k\}$ be a partition of $2g - 2$ and $\mathcal{H}(n_1, \dots, n_k)$ denote a stratum of Abelian differentials, that is, the space holomorphic 1-forms on Riemann surfaces of genus g , with zeros of degree $n_1, \dots, n_k \in \mathbb{N}$. There is a one to one correspondence between Abelian differentials and translation surfaces, surfaces which can be obtained by edge-to-edge gluing of polygons in \mathbb{R}^2 using translations only. Thus, we refer to elements of $\mathcal{H}(n_1, \dots, n_k)$ as translation surfaces. A translation surface has a canonical flat metric, the one obtained from \mathbb{R}^2 , with conical singularities of angle $2\pi(n + 1)$ at zeros of degree n of the Abelian differential.

We also consider strata $\mathcal{Q}(d_1, \dots, d_k)$ of meromorphic quadratic differentials with at most simple poles on Riemann surfaces with zeros of order d_1, \dots, d_k , $d_i \in \{-1\} \cup \mathbb{N}$ for $i = 1, \dots, k$ (in a slight abuse of vocabulary, we are considering poles as zeros of order -1) and $\sum_{i=1}^k d_i = 4g - 4$. A quadratic differential also defines a canonical flat metric with conical singularities of angle $\pi(d + 2)$ at zeros of order d .

In this paper, a quadratic differential is not the square of an Abelian differential. This condition is automatically satisfied if at least one of parameters d_j is odd.

Notation. As usual, we use “exponential” notation to denote multiple zeroes (or simple poles) of the same degree, for example $\mathcal{Q}(1, -1^5) = \mathcal{Q}(1, -1, -1, -1, -1, -1)$.

A flat surface is a Riemann surface with the flat metric corresponding to an Abelian or quadratic differential.

2.1.2. Canonical orientation double cover. One can canonically associate with every meromorphic quadratic differential q on a Riemann surface S another connected curve with an Abelian differential on it. It is the unique double covering of S (possibly ramified at singularities of q) such that the pullback of q is the square of an Abelian differential.

Notation. We denote by $\tilde{\mathcal{Q}}(d_1, \dots, d_n)$ the locus of translation surfaces consisting on the canonical orientating double cover of surfaces in the strata of half-translation surfaces $\mathcal{Q}(d_1, \dots, d_n)$.

2.1.3. Hyperelliptic surfaces, loci and components. We say that a translation surface X is a hyperelliptic surface if it corresponds to the canonical orientation double cover of a quadratic differential on a Riemann surface of genus zero. Equivalently, if $X \in \tilde{\mathcal{Q}}(d_1, \dots, d_n)$ with $\sum_{j=1}^n d_j = -4$ and, in this case, we say that $\tilde{\mathcal{Q}}(d_1, \dots, d_n)$ is an hyperelliptic locus.

There are two series of hyperelliptic loci which play a special role: for $g > 1$,

$$\begin{aligned}\tilde{\mathcal{Q}}(2g - 3, -1^{2g+1}) &\subset \mathcal{H}(2g - 2), \text{ and} \\ \tilde{\mathcal{Q}}(2g - 2, -1^{2g+2}) &\subset \mathcal{H}(g - 1, g - 1),\end{aligned}$$

In these cases, the hyperelliptic loci coincides with a connected component of the corresponding stratum (see [KZ03, §2.1]), the hyperelliptic component, which is denoted by

$$\begin{aligned}\mathcal{H}^{hyp}(2g - 2) &= \tilde{\mathcal{Q}}(2g - 3, -1^{2g+1}), \text{ and} \\ \mathcal{H}^{hyp}(g - 1, g - 1) &= \tilde{\mathcal{Q}}(2g - 2, -1^{2g+2}).\end{aligned}$$

2.1.4. Hyperelliptic involution and Weierstrass points. Every translation surface obtained as an orientation covering comes with an involution. In the case of hyperelliptic surfaces, we call it the hyperelliptic involution. The hyperelliptic involution of a hyperelliptic surface of genus g has exactly $2g + 2$ fixed points. These fixed points are called Weierstrass points. We say that a Weierstrass point is regular if it is regular for the flat metric, that is, if it is not a conical singularity. Note that regular Weierstrass points are exactly those points who projects to poles in the corresponding quadratic differential on the sphere.

Moreover, a translation surface of genus g is a hyperelliptic surface if and only if it has an involution which fixes $2g + 2$ points.

2.2. Counting problem. We are interested in the counting of closed geodesics of bounded length on translation surfaces. Together with every closed regular geodesic in a translation surface X we have a bunch of parallel closed regular geodesics. A cylinder on a flat surface is a maximal open annulus filled by isotopic simple closed regular geodesics. A cylinder C is isometric to the product of an open interval and a circle, its core curve γ_C is the geodesic projecting to the middle of the interval and its length $l(C)$ is the circumference of the circle. A saddle connection is a geodesic joining two different singularities or a singularity to itself, with no singularities in its interior. Cylinders are always bounded by parallel saddle connections.

The number of cylinders of bounded length is finite. Thus, for any $L > 0$ the following quantity is well-defined:

$$N_{area}(X, L) = \frac{1}{\text{Area}(X)} \sum_{\substack{C \subset X \\ l(C) \leq L}} \text{Area}(C),$$

where the sum is over all cylinders C in X of length bounded by L .

The following theorem is a special case of a fundamental result of Veech [Ve98], considered by Vorobets in [Vo].

Theorem (Veech). *Let ν be an ergodic $\text{SL}(2, \mathbb{R})$ -invariant probability measure on a stratum $\mathcal{H}_1(n_1, \dots, n_k)$ of Abelian differentials of area one. Then, the following ratio is constant (i.e. does not depend on the value of a positive parameter R):*

$$c_{area}(\nu) = \frac{1}{\pi R^2} \int N_{area}(X, R) d\nu.$$

This formula is the Siegel–Veech formula, and the corresponding constant $c_{area}(\nu)$ is the Siegel–Veech constant.

A fundamental result of Eskin–Mirzakhani–Mohammadi [EMM] says that every $\text{SL}(2, \mathbb{R})$ -orbit closure \mathcal{M} is an affine invariant manifold and, in particular, it is the support of an affine invariant measure $\nu_{\mathcal{M}}$ (see [EMM, EMi] for the precise definitions). For simplicity, we denote $c_{area}(\mathcal{M}) = c_{area}(\nu_{\mathcal{M}})$.

We call a configuration of cylinders on an affine invariant manifold \mathcal{M} , a continuous $\text{SL}(2, \mathbb{R})$ -equivariant application \mathcal{C} which associates to $X \in \mathcal{M}$ (or any finite cover of \mathcal{M}) a collection of cylinders in X (cf. [EMZ]). The previous discussion on the counting problem and Siegel–Veech constants applies as well in the case of configurations of cylinders and we denote by $c_{area}(\mathcal{M}, \mathcal{C})$ the corresponding Siegel–Veech constant.

Notation. For a translation surface X , we denote by $c_{area}(X)$, the Siegel–Veech constant associated to the affine invariant measure $\nu_{\mathcal{M}}$ supported on its $\mathrm{SL}(2, \mathbb{R})$ -orbit closure $\mathcal{M} = \overline{\mathrm{SL}(2, \mathbb{R})X}$. That is

$$c_{area}(X) := c_{area}(\mathcal{M}) = \frac{1}{\pi R^2} \int_{\mathcal{M}} N_{area}(Y, R) d\nu_{\mathcal{M}}(Y).$$

Similarly, for a configuration of cylinders \mathcal{C} defined on \mathcal{M} , we denote by $c_{area}(X, \mathcal{C})$, the corresponding Siegel–Veech constant, $c_{area}(X, \mathcal{C}) = c_{area}(\mathcal{M}, \mathcal{C})$.

2.3. Non-varying phenomenon. The following result summarize the non-varying phenomenon for Siegel–Veech constants observed on hyperelliptic loci by Eskin–Kontsevich–Zorich [EKZ, Theorem 3 and Lemma 1.1].

Theorem 2.1 (Eskin–Kontsevich–Zorich). *Let X be a hyperelliptic surface such that the quotient sphere belongs to $\mathcal{Q}(d_1, \dots, d_n)$. That is, $X \in \tilde{\mathcal{Q}}(d_1, \dots, d_n)$ with $\sum_{j=1}^n d_j = -4$. Then*

$$c_{area}(X) = -\frac{1}{4\pi^2} \sum_{j=1}^n d_j \frac{d_j + 4}{d_j + 2}.$$

3. THE CASE OF $\mathcal{H}^{hyp}(2g - 2)$

In this section we prove the statement of Theorem 1 in the case of the hyperelliptic component $\mathcal{H}^{hyp}(2g - 2)$, $g > 1$.

3.1. Hyperelliptic coverings. Let $X \in \mathcal{H}^{hyp}(2g - 2)$, $w_0, w_1, w_2 \in X$ be three different regular Weierstrass points¹ and $z \in X$ be the zero of order $2g - 2$ on X . Consider two saddle connections s_1, s_2 passing through w_1, w_2 respectively (and joining z to itself). In particular, s_1 and s_2 are h -invariant, where $h : X \rightarrow X$ is the hyperelliptic involution.

For $(i, j) \in \{0, 1\}^2 \setminus \{(0, 0)\}$, consider the covering X_{ij} over X defined by the subgroup of $\pi_1(X, w_0)$

$$(1) \quad \Gamma_{ij} = \{\gamma \in \pi_1(X, w_0) : \iota(\gamma, i s_1 + j s_2) \equiv_2 0\}.$$

Note that, since s_1, s_2 are closed loops, these coverings are unramified.

Lemma 3.1. *X_{ij} is hyperelliptic, for $(i, j) \in \{0, 1\}^2 \setminus \{(0, 0)\}$.*

Proof. From general covering space theory, we know that points in X_{ij} can be taken to be equivalence classes of pairs (x, ρ) , which we denote by $[x, \rho]_{ij}$, where ρ is a path joining w_0 to $x \in X$ and (x_1, ρ_1) is equivalent to (x_2, ρ_2) provided $x_1 = x_2$ and $\rho_1 \rho_2^{-1} \in \Gamma_{ij}$, where Γ_{ij} is the subgroup of $\pi_1(X, w_0)$ defining the covering X_{ij} , described by (1) above.

We define now $h_{ij} : X_{ij} \rightarrow X_{ij}$ by $h_{ij}([x, \rho]_{ij}) = [h(x), h \circ \rho]_{ij}$, where $h : X \rightarrow X$ is the hyperelliptic involution on X . Note that h_{ij} is a well defined involution which is a lift of h . It is clear from the definition that the only possible fixed points of h_{ij} are the points lying above the Weierstrass points of X . Moreover, in X_{ij} , if one of the two points above a given Weierstrass point in X is fixed by h_{ij} , then both are.

Let $w \in X$ be a Weierstrass point and $\rho : [0, 1] \rightarrow X$ be a path from w_0 to X_{11} . Consider now $\hat{\rho} := h(\rho)\rho^{-1} \in \pi_1(X, w_0)$. Then, by definition of X_{ij} (see (1) above) $[w, \rho]_{ij} \in X_{ij}$ is fixed by h_{ij} if and only if $\iota(\hat{\rho}, i s_1 + j s_2) \equiv_2 0$. Moreover, up to

¹Every hyperelliptic surface has at least four different regular Weierstrass points.

homotopy, we can suppose that $\rho|_{[0,1]}$ avoids $z, w_1, w_2 \in X$, and that ρ intersects transversally s_1, s_2 and avoids tangencies and self-intersections over s_1, s_2 . It follows that $\iota(\hat{\rho}, s_k) \equiv_2 \#\hat{\rho} \cap s_k$, $k = 1, 2$. Moreover, $\hat{\rho}$ and s_k are h -invariant and therefore, so is $\hat{\rho} \cap s_k$. Since h is an involution, the parity of $\#\hat{\rho} \cap s_k$ equals (the parity of) the number of its fixed points.

Now, since $w_0 \notin \{z, w_1, w_2\}$, we have that, for $k = 1, 2$,

- (1) if $w \notin \{z, w_k\}$, then $\hat{\rho} \cap s_k$ is an h -invariant set with no fixed points and therefore, $\iota(\hat{\rho}, s_k) \equiv_2 \#\hat{\rho} \cap s_k \equiv_2 0$;
- (2) if $w \in \{z, w_k\}$, then $\hat{\rho} \cap s_k$ is an h -invariant set with exactly one fixed point, namely X_{11} , and therefore, $\iota(\hat{\rho}, s_k) \equiv_2 \#\hat{\rho} \cap s_k \equiv_2 1$.

Thus,

- (1) the two points on X_{10} above X_{11} are fixed by h_{10} if and only if $w \notin \{z, w_1\}$;
- (2) the two points on X_{01} above X_{11} are fixed by h_{01} if and only if $w \notin \{z, w_2\}$;
- and
- (3) the two points on X_{11} above X_{11} are fixed by h_{11} if and only if $w \notin \{w_1, w_2\}$.

It follows that, for each $(i, j) \in \{0, 1\}^2 \setminus \{(0, 0)\}$, the number of points in X_{ij} , fixed by h_{ij} is twice the number of Weierstrass points in X but two. Since X is hyperelliptic of genus g , it has $2g + 2$ Weierstrass points and thus, the number of fixed points for h_{ij} is $2(2g + 2 - 2) = 4g$.

Moreover, since the coverings are regular double covers, $\chi(X_{ij}) = 2\chi(X)$. That is, $g(X_{ij}) = (2g - 2) + 1 = 2g - 1$ and h_{ij} fixes $4g = 2g(X_{ij}) + 2$ points. We conclude that X_{ij} is hyperelliptic. \square

Since X_{ij} is a regular double cover of $X \in \mathcal{H}(2g - 2)$, we have that $X_{ij} \in \mathcal{H}(2g - 2, 2g - 2)$. Furthermore, we have the following, which shall be needed latter.

Lemma 3.2. *The surfaces X_{10} and X_{01} belong to the hyperelliptic connected component $\mathcal{H}^{hyp}(2g - 2, 2g - 2) = \tilde{\mathcal{Q}}(4g - 4, -1^{4g})$, while the surface X_{11} belongs to the hyperelliptic locus $\tilde{\mathcal{Q}}(2g - 3, 2g - 3, -1^{4g-2}) \subset \mathcal{H}^{odd}(2g - 2, 2g - 2)$.*

Proof. Following the proof of Lemma 3.1, we know that the hyperelliptic involution h_{10} fixes the points in X_{10} above the Weierstrass points in X but z and w_1 . In particular, the conical singularities of X_{10} are not fixed by h_{10} and therefore $X_{10} \in \tilde{\mathcal{Q}}(4g - 4, -1^{4g}) = \mathcal{H}^{hyp}(2g - 2, 2g - 2)$. The proof for X_{01} is analogous.

Similarly, the hyperelliptic involution h_{11} fixes the points in X_{11} above the Weierstrass points in X but w_1 and w_2 . Then, the conical singularities of X_{11} are fixed by h_{11} and therefore $X_{11} \in \tilde{\mathcal{Q}}(2g - 3, 2g - 3, -1^{4g-2})$. The parity of the spin structure of surfaces in $\tilde{\mathcal{Q}}(2g - 3, 2g - 3, -1^{4g-2})$ is deduced from [KZ03, Proposition 7]. \square

3.2. Configurations and Siegel–Veech constants. We are concerned with the counting of cylinders whose core curve passes through two fixed Weierstrass points $w_1, w_2 \in X$. For a cylinder C in X , we define the profile of C to be the couple $(\iota(\gamma_C, s_k) \bmod 2)_{k=1,2} \in \{0, 1\}^2$, where γ_C is the core curve of C . We also consider \mathcal{C}_{pq} to be the configuration of cylinders in X of profile $(p, q) \in \{0, 1\}^2$.

Lemma 3.3. *The configuration \mathcal{C}_{11} coincides with the configuration of cylinders whose core curve passes through w_1 and w_2 .*

Proof. Since $X \in \mathcal{H}^{hyp}(2g-2)$, the core curve of every cylinder C in X is h -invariant and passes through exactly two different Weierstrass points². Denote by $\mathcal{W}(C)$ the set of this two Weierstrass points lying on the core curve of C . We claim that $\iota(\gamma_C, s_k) \equiv_2 \mathbf{1}_{\mathcal{W}(C)}(w_k)$. In fact, γ_C can be written as $h(\rho)\rho^{-1}$, where ρ is a geodesic path from one Weierstrass point in $\mathcal{W}(C)$ to the other. Moreover, $\rho|_{]0,1[}$ avoids every Weierstrass point, in particular z and w_k . Furthermore, since ρ and s_k are geodesics, ρ intersects transversally s_j and avoids tangencies and self-intersections over s_k . Thus, as in the proof of Lemma 3.1, it follows that $\iota(\gamma, s_k) \equiv_2 \#\gamma_C \cap s_k$ and therefore, $\iota(\gamma, s_k) \equiv_2 1$ if and only if $w_k \in \mathcal{W}(C)$, proving the claim. We conclude thus, that the core curve of C pass through w_1 and w_2 , that is, $\mathcal{W}(C) = \{w_1, w_2\}$, if and only if $\iota(\gamma, s_1) \equiv_2 \iota(\gamma, s_2) \equiv_2 1$, that is, if and only if C has profile $(1, 1)$. \square

Denote by $c_{pq} = c_{area}(X, \mathcal{C}_{pq})$. Then, it is clear that

$$c_{area}(X) = c_{00} + c_{10} + c_{01} + c_{11}.$$

Now, consider the configuration \mathcal{C}_{pq}^{ij} of cylinders C in X_{ij} such that they project to X to cylinders in \mathcal{C}_{pq} . Again, it is clear that $c_{area}(X_{ij})$ decomposes into the sum of the Siegel–Veech constants $c_{area}(X_{ij}, \mathcal{C}_{pq}^{ij})$, $(p, q) \in \{0, 1\}^2$.

The following general result relates the Siegel–Veech constants of configurations of cylinders on a double covering to the constant on the base space (see [DZ, Lemma 4.1], which is a slight generalization of [EKZ, Lemma 1.1]).

Lemma 3.4. *Let $\hat{X} \rightarrow X$ be a double covering, \mathcal{C} be a configuration of cylinders on C and $\hat{\mathcal{C}}$ be the lift of this configuration to \hat{X} . If the core curve of every cylinder in \mathcal{C} has non-trivial monodromy, then $c_{area}(\hat{X}, \hat{\mathcal{C}}) = c_{area}(X, \mathcal{C})/2$. If the core curve of every cylinder in \mathcal{C} has trivial monodromy, then $c_{area}(\hat{X}, \hat{\mathcal{C}}) = 2c_{area}(X, \mathcal{C})$.*

Note that, in our case, the monodromy of a cylinder C in X for the covering X_{ij} is, by definition, given by the intersection number $\iota(\gamma_C, is_1 + js_2) \bmod 2$. But then, profiles define the monodromy on each covering and, in particular, the relation between the Siegel–Veech constants given in Lemma 3.4 above. Thus, we have that cylinders in \mathcal{C}_{pq}^{ij} have monodromy equals to $ip + jq \bmod 2$. Using this, it is easy to verify that, by the previous lemma, we have the following:

$$\begin{aligned} c_{area}(X) &= c_{00} + c_{10} + c_{01} + c_{11}, \\ c_{area}(X_{10}) &= 2c_{00} + 2c_{10} + \frac{1}{2}c_{01} + \frac{1}{2}c_{11}, \\ c_{area}(X_{01}) &= 2c_{00} + \frac{1}{2}c_{10} + 2c_{01} + \frac{1}{2}c_{11}, \\ c_{area}(X_{11}) &= 2c_{00} + \frac{1}{2}c_{10} + \frac{1}{2}c_{01} + 2c_{11}. \end{aligned}$$

Moreover

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 1/2 & 1/2 \\ 2 & 1/2 & 2 & 1/2 \\ 2 & 1/2 & 1/2 & 2 \end{pmatrix}^{-1} = \frac{1}{3} \begin{pmatrix} -3 & 1 & 1 & 1 \\ 2 & 1 & -1 & -1 \\ 2 & -1 & 1 & -1 \\ 2 & -1 & -1 & 1 \end{pmatrix}$$

²This claim is true only for $\mathcal{H}^{hyp}(2g-2)$ and $\mathcal{H}^{hyp}(g-1, g-1)$, and this is the main reason this argument does not work on other hyperelliptic loci.

and therefore,

$$(2) \quad c_{11} = \frac{1}{3} [2c_{area}(X) - c_{area}(X_{10}) - c_{area}(X_{01}) + c_{area}(X_{11})].$$

Thus, it is enough to compute the Siegel–Veech constants for each surface and, since all of them are hyperelliptic, by Theorem 2.1, it is enough to find the strata of quadratic differentials where the quotient spheres belong. But, by Lemma 3.2, we already know this and hence,

$$c_{area}(X) = -\frac{1}{4\pi^2} \left((2g-3) \frac{2g+1}{2g-1} - 3(2g-1) \right) = \frac{g}{\pi^2} \cdot \frac{2g+1}{2g-1},$$

$$c_{area}(X_{10}) = c_{area}(X_{01}) = -\frac{1}{4\pi^2} \left((4g-4) \frac{4g}{4g-2} - 3(4g) \right) = \frac{g}{\pi^2} \cdot \frac{4g-1}{2g-1}$$

and

$$c_{area}(X_{11}) = -\frac{1}{4\pi^2} \left((2g-3) \frac{2g+1}{2g-1} 2 - 3(4g-2) \right) = \frac{1}{\pi^2} \cdot \frac{4g^2 - 4g + 3}{2g-1}.$$

Finally, plugging this in formula (2), we obtain

$$c_{11} = \frac{1}{3\pi^2} \cdot \frac{1}{2g-1} [2g(2g+1) - 2g(4g-1) + (4g^2 - 4g + 3)] = \frac{1}{\pi^2} \cdot \frac{1}{2g-1}.$$

That is, by Lemma 3.3, the Siegel–Veech constant associated to the counting of cylinders whose core curve passes through the two regular Weierstrass points w_1 and w_2 equals

$$\frac{1}{\pi^2} \cdot \frac{1}{2g-1}$$

for surfaces in $\mathcal{H}^{hyp}(2g-2)$. □

4. THE CASE OF $\mathcal{H}^{hyp}(g-1, g-1)$

In this section we prove Theorem 1 for the hyperelliptic connected components $\mathcal{H}^{hyp}(g-1, g-1)$, $g > 1$. The proof follows almost in the same way than in the case of $\mathcal{H}^{hyp}(2g-2)$ but some small details that we present below.

4.1. Hyperelliptic coverings. The construction of the hyperelliptic coverings is slightly different, in particular, the saddle connections we use to define them are no longer closed curves.

Let $X \in \mathcal{H}^{hyp}(g-1, g-1)$, $w_0, w_1, w_2 \in X$ be three different regular Weierstrass points and $z \in X$ be one of the zeros of order $g-1$ on X . Consider two saddle connections s_1, s_2 passing through w_1, w_2 respectively. In particular, s_1 and s_2 are h -invariant and goes from z to $h(z)$, where $h : X \rightarrow X$ is the hyperelliptic involution and $h(z)$ is the other zero of order $g-1$ on X .

As above, for $(i, j) \in \{0, 1\}^2 \setminus \{(0, 0)\}$, we consider the covering X_{ij} over X defined by the subgroup of $\pi_1(X, w_0)$

$$\Gamma_{ij} = \{\gamma \in \pi_1(X, w_0) : \iota(\gamma, is_1 + js_2) \equiv_2 0\}.$$

Note that, since s_1, s_2 are no longer closed loops, these coverings might be branched coverings. In fact, it is not hard to see that X_{10} and X_{01} are ramified over z and $h(z)$, while X_{11} is unramified. Anyway, they are still hyperelliptic coverings:

Lemma 4.1. *X_{ij} is hyperelliptic, for $(i, j) \in \{0, 1\}^2 \setminus \{(0, 0)\}$.*

Proof. The proof follows as in the proof of Lemma 3.1. For a Weierstrass point $w \in X$ and $\rho : [0, 1] \rightarrow X$ a path from w_0 to X_{11} , we get that $\iota(\hat{\rho}, s_k) \equiv_2 \#\hat{\rho} \cap s_k$, $k = 1, 2$, where $\hat{\rho} = h(\rho)\rho^{-1} \in \pi_1(X, w_0)$. And, for $k = 1, 2$, we have that,

- (1) if $w \neq w_k$, then $\hat{\rho} \cap s_k$ is an h -invariant set with no fixed points and therefore, $\iota(\hat{\rho}, s_k) \equiv_2 \#\hat{\rho} \cap s_k \equiv_2 0$;
- (2) if $w = w_k$, then $\hat{\rho} \cap s_k$ is an h -invariant set with exactly one fixed point, namely X_{11} , and therefore, $\iota(\hat{\rho}, s_k) \equiv_2 \#\hat{\rho} \cap s_k \equiv_2 1$.

Thus,

- (1) the two points on X_{10} above X_{11} are fixed by h_{10} if and only if $w \neq w_1$;
- (2) the two points on X_{01} above X_{11} are fixed by h_{01} if and only if $w \neq w_2$;
and
- (3) the two points on X_{11} above X_{11} are fixed by h_{11} if and only if $w \notin \{w_1, w_2\}$.

It follows that there are exactly $2(2g + 2 - 1) = 4g + 2$ fixed points for h_{10} and h_{01} . Moreover, since the coverings X_{10} and X_{01} are branched over two points, by Riemann–Hurwitz formula, $\chi(X_{10}) = \chi(X_{01}) = 2\chi(X) + 2$. That is, for $ij = 10, 01$, $g(X_{ij}) = 2g - 2 + 1 + 1 = 2g$, so h_{ij} fixes $4g + 2 = 2g(X_{ij}) + 2$ points and therefore X_{ij} is hyperelliptic. Similarly, h_{11} has $2(2g + 2 - 2) = 4g$ fixed points and since the coverings X_{11} is a regular double cover, $\chi(X_{11}) = 2\chi(X)$. That is, $g(X_{11}) = (2g - 2) + 1 = 2g - 1$. Thus h_{11} fixes $4g = 2g(X_{11}) + 2$ points and X_{11} is hyperelliptic. \square

And we have the following.

Lemma 4.2. *The surfaces X_{10} and X_{01} belong to the hyperelliptic connected component $\mathcal{H}^{hyp}(2g - 1, 2g - 1) = \tilde{\mathcal{Q}}(4g - 2, -1^{4g+2})$, while the surface X_{11} belongs to the hyperelliptic locus $\tilde{\mathcal{Q}}(2g - 2, 2g - 2, -1^{4g}) \subset \mathcal{H}(g - 1, g - 1, g - 1, g - 1)$, which is connected.*

Proof. The proof is analogous to the proof of Lemma 3.2. \square

4.2. Configurations and Siegel–Veech constants. As before, we are concerned with the counting of cylinders whose core curve passes through two fixed Weierstrass points $w_1, w_2 \in X$. For a cylinders C in X , we define the profile of C as before, that is, the couple $(\iota(\gamma_C, s_k) \bmod 2)_{k=1,2} \in \{0, 1\}^2$, where γ_C is the core curve of C ; and we consider \mathcal{C}_{pq} to be the configuration of cylinders in X of profile $(p, q) \in \{0, 1\}^2$.

Lemma 4.3. *The configuration \mathcal{C}_{11} coincides with the configuration of cylinders whose core curve passes through w_1 and w_2 .*

Proof. The proof is the same as for Lemma 3.3. \square

Denote by $c_{pq} = c_{area}(X, \mathcal{C}_{pq})$, so

$$c_{area}(X) = c_{00} + c_{10} + c_{01} + c_{11}.$$

We consider the configurations \mathcal{C}_{pq}^{ij} as in §3.2 and, applying Lemma 3.4 and the definition of profile, we obtain the same system as in the previous case. Thus, it follows that $c_{11} = [2c_{area}(X) - c_{area}(X_{10}) - c_{area}(X_{01}) + c_{area}(X_{11})]/3$.

It suffices now to compute the Siegel–Veech constants for each surface. By Theorem 2.1 and Lemma 4.2, we have that

$$c_{area}(X) = -\frac{1}{4\pi^2} \left((2g - 2) \frac{2g + 2}{2g} - 3(2g + 1) \right) = \frac{g + 1}{\pi^2} \cdot \frac{2g + 1}{2g},$$

$$c_{area}(X_{10}) = c_{area}(X_{01}) = -\frac{1}{4\pi^2} \left((4g-2) \frac{4g+2}{4g} - 3(4g+2) \right) = \frac{2g+1}{\pi^2} \cdot \frac{4g+1}{4g}$$

and

$$c_{area}(X_{11}) = -\frac{1}{4\pi^2} \left((2g-2) \frac{2g+2}{2g} - 3(4g) \right) = \frac{1}{\pi^2} \cdot \frac{2g^2+1}{g}.$$

Finally, we obtain

$$c_{11} = \frac{1}{3\pi^2} \cdot \frac{1}{4g} [4(g+1)(2g+1) - 2(2g+1)(4g+1) + 4(2g^2+1)] = \frac{1}{\pi^2} \cdot \frac{1}{2g-1}.$$

That is, by Lemma 4.3, the Siegel–Veech constant associated to the counting of cylinders whose core curve passes through the two regular Weierstrass points w_1 and w_2 equals

$$\frac{1}{\pi^2} \cdot \frac{1}{2g}$$

for surfaces in $\mathcal{H}^{hyp}(g-1, g-1)$. \square

5. COUNTEREXAMPLES

In this section we present a family of counterexamples: we exhibit hyperelliptic surfaces where the Siegel–Veech constant associated to the counting of cylinders whose core curve passes through two marked Weierstrass points does not coincide with the corresponding Siegel–Veech constant on the hyperelliptic loci where they lie.

Let X be a hyperelliptic surface in a hyperelliptic component, that is, X belongs to $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1, g-1)$, $g \geq 1$. We consider the surface X_{11} from the previous sections. By Lemma 3.2 and 4.2, we know that X_{11} belongs to the hyperelliptic locus $\tilde{\mathcal{Q}}(2g-3, 2g-3, -1^{4g-2})$ or $\tilde{\mathcal{Q}}(2g-2, 2g-2, -1^{4g})$, respectively.

Recall that in X_{11} , regular Weierstrass points are exactly those that project to regular Weierstrass points in X but w_1 and w_2 (see proof of Lemma 3.1/4.1, point (3)). For a cylinder C in X_{11} , denote by $\mathcal{W}(C)$ the set of (regular) Weierstrass points on its core curve. Thus, a cylinder C in X_{11} whose core curve passes through two regular Weierstrass points (that is, $\#\mathcal{W}(C) = 2$) projects to a cylinder in X whose core curve passes through two regular Weierstrass points different from w_1 and w_2 .

Let $\mathcal{C}_{\mathcal{W}}$ be the configuration of cylinders in X_{11} whose core curve passes through (any) two regular Weierstrass points³, and recall that \mathcal{C}_{00}^{11} denotes the configuration of cylinders in X_{11} such that their projection in X have profile $(0, 0)$ (see §3.2/4.2).

Lemma 5.1. *The configurations $\mathcal{C}_{\mathcal{W}}$ and \mathcal{C}_{00}^{11} coincide.*

Proof. The proof follows as in Lemma 3.3/4.3. In fact, let $\bar{\mathcal{C}}_{\mathcal{W}}$ denote the configuration of cylinders in X who lift to cylinders on $\mathcal{C}_{\mathcal{W}}$. By the previous discussion, $\bar{\mathcal{C}}_{\mathcal{W}}$ coincides with the configuration of cylinders in X whose core curve passes through two regular Weierstrass points different from w_1 and w_2 .

In the proof of Lemma 3.3, we showed that for any cylinder C in X we have that $\iota(\gamma_C, s_k) \equiv_2 \mathbf{1}_{\mathcal{W}(C)}(w_k)$, $k = 1, 2$. Then, since the profile of a cylinder C in X is defined as $(\iota(\gamma_C, s_k) \bmod 2)_{k=1,2}$, C has profile $(0, 0)$ if and only if $\mathbf{1}_{\mathcal{W}(C)}(w_1) = \mathbf{1}_{\mathcal{W}(C)}(w_2) = 0$. That is, if and only if its core curve passes through two regular

³Unlike the case of surfaces in hyperelliptic components, for surfaces in other hyperelliptic loci, not every core curve of a cylinder passes through Weierstrass points. However, if it passes through a Weierstrass point, it passes through exactly two of them.

Weierstrass points different from w_1 and w_2 (recall that the core curve of every cylinder in X passes through two regular Weierstrass points). Thus, $\bar{\mathcal{C}}_{\mathcal{W}} = \mathcal{C}_{00}$ and therefore, $\mathcal{C}_{\mathcal{W}} = \mathcal{C}_{00}^{11}$. \square

It follows from the previous lemma and Lemma 3.4 that $c_{area}(X_{11}, \mathcal{C}_{\mathcal{W}}) = 2c_{00}$. But we know, by the equation system it satisfies (see §3.2), that

$$c_{00} = \frac{1}{3} [-3c_{area}(X) + c_{area}(X_{10}) + c_{area}(X_{01}) + c_{area}(X_{11})].$$

We have already computed the Siegel–Veech constants for each surface. Putting all together, we obtain that

$$c_{area}(X_{11}, \mathcal{C}_{\mathcal{W}}) = \begin{cases} \frac{2g-2}{\pi^2}, & \text{if } X \in \mathcal{H}^{hyp}(2g-2), \\ \frac{2g-1}{\pi^2}, & \text{if } X \in \mathcal{H}^{hyp}(g-1, g-1). \end{cases}$$

Thus, we have computed the sum of the Siegel–Veech constants corresponding to the counting of cylinders in X_{11} whose core curve passes through two marked regular Weierstrass points, for any such marking, for a particular choice of a surfaces in the hyperelliptic loci $\tilde{\mathcal{Q}}(2g-3, 2g-3, -1^{4g-2})$ and $\tilde{\mathcal{Q}}(2g-2, 2g-2, -1^{4g})$. We shall see that these do not coincide with the generic case in such hyperelliptic loci. In fact, we have the following.

Lemma 5.2. *In the generic case, the value of the Siegel–Veech constants is*

$$c_{area}(\mathcal{M}, \mathcal{C}_{\mathcal{W}}) = \begin{cases} \frac{2g-1}{\pi^2}, & \text{if } \mathcal{M} = \tilde{\mathcal{Q}}(2g-3, 2g-3, -1^{4g-2}), \\ \frac{2g}{\pi^2}, & \text{if } \mathcal{M} = \tilde{\mathcal{Q}}(2g-2, 2g-2, -1^{4g}). \end{cases}$$

Proof. By a result of Athreya–Eskin–Zorich ([AEZ, Corollary 4.7]), the generic classical Siegel–Veech constant associated to the counting of cylinders in \mathbb{CP}^1 (with a meromorphic quadratic differential with at most simple poles) bounded by a saddle connection joining two marked poles equals $1/2\pi^2$ (whichever stratum of quadratic differentials in genus zero). Note that, these cylinders correspond exactly to the projection of cylinders in the orientation double cover whose core curve passes through two marked regular Weierstrass points.

However, we are interested in the *area* Siegel–Veech constant. For configurations \mathcal{C} of cylinders in strata $\mathcal{L} = \mathcal{Q}(d_1, \dots, d_k)$ of quadratic differentials on \mathbb{CP}^1 (that is, $\sum_{j=1}^k d_k = -4$), there exist a relation between the classical Siegel–Veech constant $c(\mathcal{L}, \mathcal{C})$ and the area Siegel–Veech constant $c_{area}(\mathcal{L}, \mathcal{C})$, namely

$$c_{area}(\mathcal{L}, \mathcal{C}) = \frac{1}{k-3} c(\mathcal{L}, \mathcal{C}).$$

This is a consequence of a generalization of Vorobets formula [Vo, Theorem 1.6(b)], proved by Athreya–Eskin–Zorich [AEZ, Proposition 4.9] for any configuration of cylinders on any strata of quadratic differentials on \mathbb{CP}^1 . Moreover, by [EKZ, Lemma 1.1] (cf. Lemma 3.4), we have that the corresponding Siegel–Veech constant in the hyperelliptic locus, say $c_{area}(\tilde{\mathcal{L}}, \tilde{\mathcal{C}})$, is twice this value,

$$c_{area}(\tilde{\mathcal{L}}, \tilde{\mathcal{C}}) = 2c_{area}(\mathcal{C}) = \frac{2}{k-3} c(\mathcal{L}, \mathcal{C}).$$

Putting all together, in the case of $\mathcal{M}_1 = \tilde{Q}(2g-3, 2g-3, -1^{4g-2})$ we get

$$c_{area}(\mathcal{M}_1, \mathcal{C}_W) = \frac{2}{4g-3} \binom{4g-2}{2} \frac{1}{2\pi^2} = \frac{2g-1}{\pi^2},$$

where the binomial coefficient stands for all possible choices of two regular Weierstrass points, which correspond to the poles of the quadratic differential on \mathbb{CP}^1 .

Similarly, in the case of $\mathcal{M}_2 = \tilde{Q}(2g-2, 2g-2, -1^{4g})$, we get

$$c_{area}(\mathcal{M}_2, \mathcal{C}_W) = \frac{2}{4g-1} \binom{4g}{2} \frac{1}{2\pi^2} = \frac{2g}{\pi^2}.$$

□

Thus, the Siegel–Veech constant for the counting of cylinders whose core curve passes through two marked regular Weierstrass points cannot be non-varying in $\tilde{Q}(2g-3, 2g-3, -1^{4g-2})$ or $\tilde{Q}(2g-2, 2g-2, -1^{4g})$, as the sum of all of them is not.

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